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THE APPROPRIATE COROTATIONAL RATE, EXACT FORMULA FOR THE PLASTIC SPIN AND CONSTITUTIVE MODEL FOR FINITE ELASTOPLASTICITY

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Abstract—The exact formulae for the plastic and the elastic spin referred to the deformed configuration are derived, where the plastic spin is a function of the plastic strain rate and the elastic spin a function of the elastic strain rate. With these exact formulae we determine the macroscopic substructure spin that allows us to define the appropriate corotational rate for finite elastoplasticity.

Plastic, elastic and substructure spin are considered and simplified for various sub-classes of restricted elastic–plastic strains. It is shown that for the special cases of rigid-plasticity and hypo-elasticity the proposed corotational rate is identical with the Green–Naghdi rate, while the Zaremba–Jaumann rate yields a good approximation for moderately large strains.

To compare our exact plastic spin formula with the constitutive assumption for the plastic spin introduced by Dafalias and others, we simplify our result for small elastic–moderate plastic strains and introduce a simplest evolution law for kinematic hardening leading to the Dafalias formula and to an exact determination of its unknown coefficient. It is also shown that contrary to statements in the literature the plastic spin is not zero for vanishing kinematic hardening.

For isotropic-elastic material with induced plastic flow undergoing isotropic and kinematic hardening constitutive and evolution laws are proposed. Elastic and plastic Lagrangean and Eulerian logarithmic strain measures are introduced and their material time derivatives and corotational rates, respectively, are considered. Finally, the elastic–plastic tangent operator is derived.

The presented theory is implemented in a solution algorithm and numerically applied to the simple shear problem for finite elastic–finite plastic strains as well as for sub-classes of restricted strains. The results are compared with those of the literature and with those obtained by using other corotational rates.

1. INTRODUCTION

In finite elastoplasticity the basic kinematic assumption is the multiplicative decomposition of the total deformation gradient $\mathbf{F}$ into an elastic, $\mathbf{F}^e$, and a plastic, $\mathbf{F}^p$, part, $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$, introduced by Bilby *et al.* (1957), Kröner (1960) and Lee (1969). It is well-known that within a macro-theory of finite elastoplasticity the corresponding elastic and plastic rotations cannot be determined uniquely. A unique decomposition of $\mathbf{F}$ without any assumption concerning the elastic and plastic rotations was first presented by Nemat-Nasser (1990). In Schieck and Stumpf (1993) and Stumpf and Schieck (1994) we proposed an alternative objective decomposition, $\mathbf{F} = \mathbf{Q}\mathring{\mathbf{U}}^e\mathbf{U}^p$, where $\mathbf{U}^p$ is the Lagrangean plastic stretch, $\mathring{\mathbf{U}}^e$ a back-rotated Lagrangean-type elastic stretch and $\mathbf{Q}$ a uniquely defined rotation tensor. Using this decomposition we derived the Lagrangean objective kinematics of two superposed finite elastic–plastic deformations. By restricting the superposed deformation to comprise only moderately large strains and by introducing elastic and plastic logarithmic strain measures as basic kinematical variables a Lagrangean-type solution algorithm for finite elastic–plastic strains was obtained. No assumptions were needed concerning the elastic and plastic rotation or the plastic spin. Dashner (1993) intended to show in his contribution that our results can be obtained also by simplifying our exact kinematics. From eqn $(8)_3$, Dashner (1993), it can be seen that his "elastic" deformation gradient $\mathbf{F}_e$ depends also on the plastic stretch $\bar{\mathbf{U}}_p$. This is also the case in his further definitions of the "elastic" stretch tensor $\overset{+}{\mathbf{V}}_e$, logarithmic strain tensor $\bar{\mathbf{H}}_e$ and also in his strain energy density ψ, while in our corresponding formulae all elastic measures are purely elastic quantities. It can be shown by estimations that the influence of the plastic stretch $\overset{+}{\mathbf{U}}_p$ in Dashner's formulae is cancelled only if the deformation is restricted to moderately large total elastic and moderately large superposed plastic strains, a special case of our general result. This class of deformations was first investigated by Etorovic and Bathe (1990). In Schieck and Stumpf (1993) we did not consider an appropriate formulation of evolution laws for kinematic hardening and also not the influence of the plastic spin on the hardening, which is the subject of the present paper.

By analysing the plastic deformation of the simple shear problem with kinematic hardening using the Zaremba–Jaumann corotational rate Dienes (1979) observed an oscillatory stress response, which is physically not acceptable. His paper initiated a broad discussion in the literature concerning the appropriate choice of a corotational rate in finite elastoplasticity [e. g. Atluri (1984), Haupt (1985), Szabo and Balla (1989), Xia and Ellyin (1993), Zbib (1993)]. Using some ideas of Mandel (1971) and Kratochvil (1971), Dafalias (1983), (1984), Loret (1983) and Onat (1984) introduced an additional constitutive equation for the plastic spin to obtain an appropriate corotational rate for kinematic hardening. They presented a general plastic spin formula with unknown coefficients derived by using the representation theorem. This result seemed to show rigorously [Van der Giessen (1991), p. 369)] that the plastic spin vanishes for isotropic plastic material behaviour, which was first mentioned by Mandel (1971) and Kratochvil (1971) and later generally accepted and also used in proposed solution algorithms of finite elastoplasticity. A simplest approximation of this general constitutive plastic spin formula for small elastic – finite plastic strains and kinematic hardening was derived by Dafalias (1985a, b) and investigated by many authors [e.g. Zbib and Aifantis (1988), Raniecki and Samanta (1989), Dafalias (1990), Van der Giessen (1991), Paulun and Pecherski (1985), (1992)]. Contrary to these papers it was shown independently in Stumpf and Badur (1990) and Nemat-Nasser (1990) that the plastic spin is a kinematic function depending on the plastic strain rate and that therefore additional constitutive equations for the plastic spin cannot be introduced.

Simo (1988) presented a concept for finite elastoplasticity based on the multiplicative decomposition of the deformation gradient, $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$, using as objective constitutive rates Lie derivatives. Models of this type are considered and applied also in Duszek-Perzyna and Perzyna (1993), Stumpf (1993) and Le and Stumpf (1993). In the latter paper equivalent sets of balance equations and entropy production inequality are presented in spatial, referential and intermediate description. The entropy production inequality formulated with respect to the intermediate configuration together with an assumption about the functional form of the free energy are used to substantiate new constitutive equations. In this theory work-conjugate to the plastic strain rate tensor is the plastic Eshelby tensor. Isotropic and kinematic hardening laws are not given. Within all the above mentioned macro-theories of finite elastoplasticity it is not possible to determine uniquely the stress-free intermediate configuration.

In Le and Stumpf (1994), (1995a) a model of finite elastoplasticity with microstructure and a framework for a nonlinear dislocation theory, respectively, are presented based on the multiplicative decomposition of the deformation gradient. In this theory dislocations with corresponding couple stresses and the dislocation motion are taken into account. The stress-free intermediate configuration is treated as crystal reference, where the internal energy per unit crystal volume is a function of the elastic deformation and the torsion tensor. The torsion describes the dislocation density defined by the dynamics of the plastic flow. In Le and Stumpf (1994), (1995b) it is shown that within this micro-theory the plastic deformation gradient $\mathbf{F}^p$ and the plastic rotation $\mathbf{R}^p$, respectively, can be determined uniquely.

Based on the unique decomposition of Schieck and Stumpf (1993) and Stumpf and Schieck (1994), $\mathbf{F} = \mathbf{Q}\mathring{\mathbf{U}}^e\mathbf{U}^p$, we derive in the present paper the appropriate co-rotational rate valid for the whole range of finite elastoplasticity including rigid-plastic and hypoelastic deformations. Furthermore we formulate the Eulerian constitutive and evolution equations for isotropic elastic material with induced plastic flow undergoing isotropic and kinematic hardening.

In Section 2 we shortly discuss the requirements of frame indifference according to

Truesdell and Noll (1965). In Section 3 we construct the corotational rate by using the spin $\boldsymbol{\Omega} = \dot{\mathbf{Q}}\mathbf{Q}^{-1}$ referred to the actual deformed configuration and defined by $\boldsymbol{\Omega} = \mathbf{w} - \mathbf{w}^{e} - \mathbf{w}^{p}$, where $\mathbf{w}$ is the material spin, $\mathbf{w}^{e}$ the elastic and $\mathbf{w}^{p}$ the plastic spin. We present the exact formula for the elastic spin $\mathbf{w}^{e}$ as function of the elastic strain rate $\mathbf{d}^{e}$ and the exact formula for the plastic spin $\mathbf{w}^{p}$ as function of the plastic strain rate $\mathbf{d}^{p}$ valid for the whole range of finite elastic and finite plastic strains (Section 3). The result shows rigorously that the plastic spin vanishes only if the elastic stretch, the plastic stretch and the plastic strain rate are coaxial. For small elastic strains the plastic stretch and the plastic strain rate must be coaxial, which is in general not the case even without any hardening. Neglecting the plastic spin $\mathbf{w}^{p}$ for small elastic–finite plastic deformation means that the appropriate corotational rate with the spin $\boldsymbol{\Omega}$ is replaced by the Zaremba–Jaumann rate with the material spin $\mathbf{w}$ leading to wrong results and the well-known oscillations. In Section 3 we present furthermore the relation between the spin $\boldsymbol{\Omega} = \dot{\mathbf{Q}}\mathbf{Q}^{-1}$ and the spin $\boldsymbol{\Omega}_{R} = \dot{\mathbf{R}}\mathbf{R}^{-1}$, where the rotation tensor $\mathbf{R}$ follows from the polar decomposition theorem, $\mathbf{F} = \mathbf{RU}$. This relation enables a comparison between the general corotational rate proposed in this paper and the Green–Naghdi rate [Green and Naghdi, (1965)].

In Section 4 we consider elastic, $\mathbf{w}^{e}$, plastic, $\mathbf{w}^{p}$, and the proposed spin $\boldsymbol{\Omega}$ for restricted classes of elastic and/or plastic strains: rigid-plastic, hypoelastic, small elastic – finite plastic, moderate elastic – moderate plastic and finite elastic – small plastic strains. It is shown that our corotational rate valid for the whole range of finite elastic–plastic deformations coincides with the Green–Naghdi rate for rigid-plasticity and hypoelasticity, while the Zaremba–Jaumann rate yields a good approximation within moderate elastic and moderate plastic strains.

To compare our exact plastic spin formula with the constitutive assumption for the plastic spin introduced by Dafalias (1985a,b) for kinematic hardening, we simplify in Section 5 our result for small elastic–moderate plastic strains and introduce a simplest evolution law for kinematic hardening leading to the Dafalias formula and to an exact determination of its unknown coefficient. It is also shown that contrary to statements of the literature the plastic spin is not zero for vanishing kinematic hardening.

In Section 6 we present elastic and plastic constitutive and evolution equations. Taking into account results of Anand (1979, 1986) we assume for isotropic elastic material subject to moderate elastic strains the existence of an elastic potential as function of the logarithmic back-rotated elastic strain tensor $\mathring{\mathbf{H}}^{e}$ and the fourth-order elasticity tensor of the linear theory. Hyper- and hypoelastic constitutive equations are given using the material time derivative and the corotational rate, respectively, of the Lagrangean and Eulerian logarithmic strain tensors. The evolution law for the backstress tensor is assumed of "hyperplastic" and "hypoplastic" type. By "hyperplastic" we mean that there exists a potential for the backstress tensor, and by "hypoplastic" we denote the corotational rate formulation of the backstress tensor. At the end of Section 6 the elastic–plastic tangential operator is given, which can easily be implemented in a general solution algorithm.

In Section 7 the concept of this paper is applied to analyse the simple shear problem for finite elastic–finite plastic strains as well as for various subclasses of restricted strains. The results are compared with those of the literature as well as with those obtained by using other corotational rates. To show that the corotational rate proposed in this paper is appropriate for the whole range of finite elastic–plastic deformation we analyse numerically also a closed deformation cycle for hyperelastic and hypoelastic material behaviour. The results show that in the case of hypoelasticity the residual stresses after a closed cycle vanish only if an elastic potential exists and if our corotational rate or the Green–Naghdi rate, which are identical in this case, are used, while the Zaremba–Jaumann rate leads to non-vanishing residual stresses [see also Kojic and Bathe (1987)].

2. INVARIANCE REQUIREMENTS

Within classical continuum mechanics having the Newtonian space–time as the basic underlying concept the constitutive relations must satisfy the principle of frame-indifference (Truesdell and Noll, 1965; Wang and Truesdell, 1977; Ogden, 1984). Accordingly, by way of preparation for the subsequent considerations we collect in this chapter the relevant transformation rules for various quantities implied by the change of a frame of reference. To this end let us recall that the Newtonian space–time may be represented isometrically by a product space $\mathscr{E} \times \mathscr{T}$, where the three-dimensional Euclidean point space $\mathscr{E}$ is referred to as instantaneous physical space or space of places and $\mathscr{T} \subset \mathbb{R}$ as time interval. The representation of space–time by the product space $\mathscr{E} \times \mathscr{T}$ is called framing (a frame of reference or an observer) which provides a background, relative to which the motion and deformation of the body are specified. A change of frame of reference is a time dependent family of distance preserving invertible maps of the space–time into itself. In particular, if $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ and $\mathbf{x}^+ = \mathbf{x}^+(\mathbf{X}, t^+)$ describe the same motion of the material body $\mathscr{B}$ relative to two frames of reference, then

$$\mathbf{x}^+ = \mathbf{c}(t) + \mathbf{O}(t)\mathbf{x}, \quad t^+ = c + t, \tag{1}$$

where $\mathbf{c}(t) \in E$ is a time dependent vector, $\mathbf{O}(t) \in O(3)$ is a time dependent orthogonal tensor and $c \in \mathbb{R}$ is a constant. A change of frame of reference induces definite rules of transformations for defined quantities. For primitive quantities such rules must be specified by axioms. In either case scalar, vector or 2nd-order tensor fields are said to be frame-indifferent, if under the change of frame of reference they transform according to the rules

$$\begin{aligned} f^+(\mathbf{x}^+, t^+) &= f(\mathbf{x}, t), \\ \mathbf{v}^+(\mathbf{x}^+, t^+) &= \mathbf{O}(t)\mathbf{v}(\mathbf{x}, t), \\ \mathbf{T}^+(\mathbf{x}^+, t^+) &= \mathbf{O}(t)\mathbf{T}(\mathbf{x}, t)\,\mathbf{O}^{\mathrm{T}}(t). \end{aligned} \tag{2}$$

Given a motion $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ of the body we use the standard notation for the deformation gradient, the spatial gradient of the velocity field and their decompositions

$$\mathbf{F} = \nabla\mathbf{x} = \mathbf{RU} = \mathbf{VR}, \tag{3}$$

$$\mathbf{l} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \operatorname{grad}\dot{\mathbf{x}} = \mathbf{d} + \mathbf{w}, \quad \mathbf{d} = \tfrac{1}{2}(\mathbf{l} + \mathbf{l}^{\mathrm{T}}), \quad \mathbf{w} = \tfrac{1}{2}(\mathbf{l} - \mathbf{l}^{\mathrm{T}}), \tag{4}$$

where $\dot{\mathbf{x}}$ denotes the spatial velocity field. By virtue of (1) we easily find that under change of the frame of reference the respective quantities transform according to the rules

$$\begin{aligned} &\overset{+}{\mathbf{F}} = \mathbf{OF}, \\ &(\mathring{\mathbf{U}}^{e})^+ = \mathring{\mathbf{U}}^{e}, \quad (\mathbf{U}^{p})^+ = \mathbf{U}^{p}, \\ &\overset{+}{\mathbf{Q}} = \mathbf{OQ}, \quad \overset{+}{\mathbf{R}} = \mathbf{OR}, \\ &\overset{+}{\mathbf{V}}{}^{e} = \mathbf{OV}^{e}\mathbf{O}^{\mathrm{T}}, \quad \overset{+}{\mathbf{V}}{}^{p} = \mathbf{OV}^{p}\mathbf{O}^{\mathrm{T}}, \\ &(\overset{\vee}{\mathbf{V}}{}^{e})^+ = \mathbf{O}\overset{\vee}{\mathbf{V}}{}^{e}\mathbf{O}^{\mathrm{T}}, \quad (\overset{\vee}{\mathbf{V}}{}^{p})^+ = \mathbf{O}\overset{\vee}{\mathbf{V}}{}^{p}\mathbf{O}^{\mathrm{T}}, \\ &\overset{+}{\mathbf{l}} = \dot{\mathbf{O}}\mathbf{O}^{\mathrm{T}} + \mathbf{OlO}^{\mathrm{T}}, \\ &\overset{+}{\mathbf{d}} = \mathbf{OdO}^{\mathrm{T}}, \quad \overset{+}{\mathbf{w}} = \dot{\mathbf{O}}\mathbf{O}^{\mathrm{T}} + \mathbf{OwO}^{\mathrm{T}}, \\ &\overset{+}{\mathbf{d}}{}^{e} = \mathbf{Od}^{e}\mathbf{O}^{\mathrm{T}}, \quad \overset{+}{\mathbf{w}}{}^{e} = \mathbf{Ow}^{e}\mathbf{O}^{\mathrm{T}}, \\ &\overset{+}{\mathbf{d}}{}^{p} = \mathbf{Od}^{p}\mathbf{O}^{\mathrm{T}}, \quad \overset{+}{\mathbf{w}}{}^{p} = \mathbf{Ow}^{p}\mathbf{O}^{\mathrm{T}}, \\ &\overset{+}{\mathbf{\Omega}} = \dot{\mathbf{O}}\mathbf{O}^{\mathrm{T}} + \mathbf{O\Omega O}^{\mathrm{T}}, \\ &\overset{+}{W}(\ln \overset{+}{\mathring{\mathbf{U}}}{}^{e}) = W(\ln \mathring{\mathbf{U}}^{e}), \\ &\overset{+}{\boldsymbol{\tau}} = \mathbf{O}\boldsymbol{\tau}\mathbf{O}^{\mathrm{T}}. \end{aligned} \tag{5}$$

The principle of frame-indifference requires that the constitutive equations of Section 6 are frame-indifferent. With (1)–(5) it is a simple matter to confirm that the constitutive model presented in this paper is frame-indifferent (objective).

3. KINEMATICS OF FINITE ELASTOPLASTICITY

The basic kinematic assumption of finite elasto-plasticity is the decomposition of the deformation gradient $\mathbf{F}$ into an elastic, $\mathbf{F}^e$, and a plastic, $\mathbf{F}^p$, part,

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p. \tag{6}$$

It is well-known that within a macro-theory of elasto-plasticity this decomposition is unique only up to an arbitrary rotation. Therefore, in the frame of Cauchy continua the intermediate configuration associated with (6) cannot be defined uniquely.

Alternatively to (6), we introduced in Schieck and Stumpf (1993) and Stumpf and Schieck (1994) a unique decomposition of the deformation gradient, which we will use in the present paper as point of departure to derive the exact formulae for the plastic spin as function of the plastic strain rate, for the elastic spin as function of the elastic strain rate, and for the spin $\mathbf{\Omega}$.

Applying the polar decomposition theorem to $\mathbf{F}^e$ and $\mathbf{F}^p$, we have

$$\mathbf{F}^e = \mathbf{R}^e\mathbf{U}^e, \quad \mathbf{F}^p = \mathbf{R}^p\mathbf{U}^p \tag{7}$$

with $\mathbf{U}^e$, $\mathbf{U}^p$ the Lagrangean-type elastic and plastic stretches and $\mathbf{R}^e$, $\mathbf{R}^p$ the elastic and plastic rotations. Defining a back-rotated elastic stretch tensor $\mathring{\mathbf{U}}^e$ and the composition $\mathbf{Q}$ of the plastic and elastic rotation tensors

$$\mathring{\mathbf{U}}^e := \mathbf{R}^{pT}\mathbf{U}^e\mathbf{R}^p, \quad \mathbf{Q} := \mathbf{R}^e\mathbf{R}^p, \tag{8}$$

we obtain the unique decomposition

$$\mathbf{F} = \mathbf{Q}\mathring{\mathbf{U}}^e\mathbf{U}^p. \tag{9}$$

On the other side the polar decomposition theorem yields for the total deformation gradient $\mathbf{F}$,

$$\mathbf{F} = \mathbf{R}\mathbf{U}, \tag{10}$$

with the total rotation $\mathbf{R}$ and the total stretch $\mathbf{U}$. With (9) and (10) we can derive a decomposition of the total stretch $\mathbf{U}$ into elastic and plastic stretches $\mathring{\mathbf{U}}^e$, $\mathbf{U}^p$,

$$\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}} = \sqrt{\mathbf{U}^p\mathring{\mathbf{U}}^{e2}\mathbf{U}^p}. \tag{11}$$

According to the decomposition (9) the undeformed configuration $\mathring{\mathscr{B}}$ is first plastically stretched by $\mathbf{U}^p$ into $\mathring{\mathscr{B}}^p$, followed by an elastic stretch $\mathring{\mathbf{U}}^e$ leading to $\mathring{\mathscr{B}}^{ep}$ and finally rotated by $\mathbf{Q}$ to the final configuration $\mathscr{B}$. Here $\mathring{\mathscr{B}}^p$ and $\mathring{\mathscr{B}}^{ep}$ denote local references.

In the macro-model of finite elastoplasticity with a description of the hardening phenomena by two macro-variables, a scalar-valued variable for isotropic hardening and the backstress tensor for kinematic hardening, we have to refer the backstress tensor to the reference $\mathring{\mathscr{B}}^p$. Then the rotation tensor $\mathbf{Q}$ can be considered as macro-substructure rotation and its spin

$$\mathbf{\Omega} = \dot{\mathbf{Q}}\mathbf{Q}^{-1} \tag{12}$$

as macro-substructure spin, which is crucial to formulate adequate Eulerian constitutive and evolution equations in finite elastoplasticity, as will be shown in the following sections.

Besides the Lagrangean stretches $\mathbf{U}^p$, $\mathring{\mathbf{U}}^e$, we introduce the Eulerian-type stretches $\mathbf{V}^e$, $\mathbf{V}^p$ by push-forward with $\mathbf{Q}$,

$$\mathbf{V}^e := \mathbf{Q}\mathring{\mathbf{U}}^e\mathbf{Q}^T, \quad \mathbf{V}^p := \mathbf{Q}\mathbf{U}^p\mathbf{Q}^T, \tag{13}$$

and define their objective corotational rates as follows,

$$\begin{aligned} \overset{\nabla}{\mathbf{V}}{}^e &:= \mathbf{Q}\dot{\mathring{\mathbf{U}}}{}^e\mathbf{Q}^T = \dot{\mathbf{V}}^e - \mathbf{\Omega}\mathbf{V}^e + \mathbf{V}^e\mathbf{\Omega} \\ \overset{\nabla}{\mathbf{V}}{}^p &:= \mathbf{Q}\dot{\mathbf{U}}^p\mathbf{Q}^T = \dot{\mathbf{V}}^p - \mathbf{\Omega}\mathbf{V}^p + \mathbf{V}^p\mathbf{\Omega}, \end{aligned} \tag{14}$$

with $\mathbf{\Omega}$ according to (12).

Differentiating eqn (9) and using (14) we obtain the velocity gradient $\mathbf{l}$ referred to the actual configuration in the form

$$\mathbf{l} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \mathbf{\Omega} + \overset{\nabla}{\mathbf{V}}{}^e\mathbf{V}^{e-1} + \mathbf{V}^e\overset{\nabla}{\mathbf{V}}{}^p\mathbf{V}^{p-1}\mathbf{V}^{e-1}. \tag{15}$$

Considering the symmetric and the screw-symmetric parts of the velocity gradient $\mathbf{l}$ and splitting them into their elastic and plastic contributions we can derive the total, elastic and plastic deformation rates $\mathbf{d}$, $\mathbf{d}^e$, $\mathbf{d}^p$,

$$\mathbf{d} = \tfrac{1}{2}(\mathbf{l} + \mathbf{l}^T) = \mathbf{d}^e + \mathbf{d}^p \tag{16}$$

with

$$\mathbf{d}^e = \tfrac{1}{2}(\overset{\nabla}{\mathbf{V}}{}^e\mathbf{V}^{e-1} + \mathbf{V}^{e-1}\overset{\nabla}{\mathbf{V}}{}^e), \tag{17}$$

$$\mathbf{d}^p = \tfrac{1}{2}(\mathbf{V}^e\overset{\nabla}{\mathbf{V}}{}^p\mathbf{V}^{p-1}\mathbf{V}^{e-1} + \mathbf{V}^{e-1}\mathbf{V}^{p-1}\overset{\nabla}{\mathbf{V}}{}^p\mathbf{V}^e), \tag{18}$$

and the total, elastic and plastic spins $\mathbf{w}$, $\mathbf{w}^e$, $\mathbf{w}^p$,

$$\mathbf{w} = \tfrac{1}{2}(\mathbf{l} - \mathbf{l}^T) = \mathbf{\Omega} + \mathbf{w}^e + \mathbf{w}^p, \tag{19}$$

with

$$\mathbf{w}^e = \tfrac{1}{2}(\overset{\nabla}{\mathbf{V}}{}^e\mathbf{V}^{e-1} - \mathbf{V}^{e-1}\overset{\nabla}{\mathbf{V}}{}^e), \tag{20}$$

$$\mathbf{w}^p = \tfrac{1}{2}(\mathbf{V}^e\overset{\nabla}{\mathbf{V}}{}^p\mathbf{V}^{p-1}\mathbf{V}^{e-1} - \mathbf{V}^{e-1}\mathbf{V}^{p-1}\overset{\nabla}{\mathbf{V}}{}^p\mathbf{V}^e). \tag{21}$$

The spin $\mathbf{\Omega}$ follows from (19),

$$\mathbf{\Omega} = \mathbf{w} - \mathbf{w}^e - \mathbf{w}^p. \tag{22}$$

To derive a closed form representation of the elastic spin $\mathbf{w}^e$ and of the plastic spin $\mathbf{w}^p$ we introduce fourth order tensors $\mathbb{A}^e$ and $\mathbb{A}^p$ such that eqns (17), (18) can be rewritten as follows,

$$\mathbf{d}^e = \mathbb{A}^e \cdot \overset{\nabla}{\mathbf{V}}{}^e, \quad \mathbf{d}^p = \mathbb{A}^p \cdot \overset{\nabla}{\mathbf{V}}{}^p, \tag{23}$$

where a dot denotes the tensor product with contraction of two indices, if index notation is used. The tensors $\mathbb{A}^e$ and $\mathbb{A}^p$ are given explicitly in Appendix A. Inverting eqns (23) and introducing them into (20), (21), we obtain the exact formulae for the elastic and the plastic spin,

$$\mathbf{w}^e = \tfrac{1}{2}((\mathbf{d}^e \cdot \mathbb{A}^{e\ \mathrm{T}})\mathbf{V}^{e-1} - \mathbf{V}^{e-1}(\mathbb{A}^{e-1} \cdot \mathbf{d}^e)), \tag{24}$$

$$\mathbf{w}^p = \tfrac{1}{2}(\mathbf{V}^e(\mathbf{d}^p \cdot \mathbb{A}^{p-\mathrm{T}})\mathbf{V}^{p-1}\mathbf{V}^{e-1} - \mathbf{V}^{e-1}\mathbf{V}^{p-1}(\mathbb{A}^{p-1} \cdot \mathbf{d}^p)\mathbf{V}^e). \tag{25}$$

It can be seen that the elastic spin $\mathbf{w}^e$ is uniquely defined by the elastic strain rate $\mathbf{d}^e$ and the elastic stretch $\mathbf{V}^e$ and that the plastic spin $\mathbf{w}^p$ is uniquely defined by the plastic strain rate $\mathbf{d}^p$, the plastic stretch $\mathbf{V}^p$, and the elastic stretch $\mathbf{V}^e$. This shows that elastic and plastic spin can be determined exactly by kinematical considerations only and that additional constitutive assumptions for the plastic spin are inadequate. Furthermore, we can see from eqns (18) and (21) immediately, that the plastic spin $\mathbf{w}^p$ vanishes only if $\mathbf{V}^e$, $\mathbf{V}^p$ and $\mathbf{d}^p$ are coaxial. For small elastic strains the influence of $\mathbf{V}^e$ can be neglected [see eqn (44)], and the plastic spin $\mathbf{w}^p$ is zero only if $\mathbf{V}^p$ and $\mathbf{d}^p$ are coaxial, which is in general not the case. This result corrects the usual assumption of finite elastoplasticity that the plastic spin $\mathbf{w}^p$ is zero for vanishing kinematic hardening.

Introducing (24), (25) into (22) we obtain the exact formula for the spin $\mathbf{\Omega}$.

Besides the spins $\mathbf{w}$, $\mathbf{w}^e$, $\mathbf{w}^p$, $\mathbf{\Omega}$ connected by eqn (22) we have to consider also the spin $\mathbf{\Omega}_R$,

$$\mathbf{\Omega}_R = \dot{\mathbf{R}}\mathbf{R}^{-1}, \tag{26}$$

where $\mathbf{R}$ is the rotation tensor following from the polar decomposition (10) of the total deformation gradient $\mathbf{F}$. The spin (26) is used in the well-known Green–Naghdi rate (Green and Naghdi, 1965).

To derive a correlation between the spins $\mathbf{\Omega}_R$ and $\mathbf{\Omega}$ according to (12) let us consider the composition of the two stretches $\mathbf{U}^p$ and $\mathring{\mathbf{U}}^e$, which is in general not symmetric,

$$\mathring{\mathbf{F}} := \mathring{\mathbf{U}}^e\mathbf{U}^p. \tag{27}$$

Applying the polar decomposition theorem to $\mathring{\mathbf{F}}$ we obtain (see also Schieck and Stumpf, 1993),

$$\mathring{\mathbf{F}} = \overset{*}{\mathbf{R}}\overset{*}{\mathbf{U}}, \tag{28}$$

$$\overset{*}{\mathbf{U}} = \sqrt{\mathring{\mathbf{F}}^{\mathrm{T}}\mathring{\mathbf{F}}} = \sqrt{\mathbf{U}^p\mathring{\mathbf{U}}^{e2}\mathbf{U}^p} = \mathbf{U}, \tag{29}$$

$$\overset{*}{\mathbf{R}} = \mathring{\mathbf{U}}^e\mathbf{U}^p\mathbf{U}^{-1}. \tag{30}$$

Introducing (28) into (9) and comparing (9) with (10) the total rotation tensor $\mathbf{R}$ follows as composition of the rotation $\overset{*}{\mathbf{R}}$ and the substructure rotation $\mathbf{Q}$,

$$\mathbf{R} = \mathbf{Q}\overset{*}{\mathbf{R}}. \tag{31}$$

By differentiation of (31) we obtain

$$\mathbf{\Omega}_R = \mathbf{\Omega} + \mathbf{Q}\overset{*}{\mathbf{\Omega}}\mathbf{Q}^{\mathrm{T}}, \quad \overset{*}{\mathbf{\Omega}} = \dot{\overset{*}{\mathbf{R}}}\overset{*}{\mathbf{R}}^{-1}, \tag{32}$$

this means that the spin $\mathbf{\Omega}_R$ is equal to the sum of the spin $\mathbf{\Omega}$ and the push-forward with $\mathbf{Q}$ of the spin $(32)_2$.

The following corotational rates are well-known in hypoelasticity and finite elastoplasticity, the Zaremba–Jaumann (ZJ) rate, constructed with the material spin $\mathbf{w}$, and the Green–Naghdi (GN) rate, constructed with the spin $\mathbf{\Omega}_{\mathrm{R}}$ according to (26),

$$(\overset{\Delta}{\ }) = (\dot{\ }) - \mathbf{w}(\ ) + (\ )\mathbf{w}, \quad \text{(ZJ)} \tag{33}$$

$$(\overset{\square}{\ }) = (\dot{\ }) - \mathbf{\Omega}_{\mathrm{R}}(\ ) + (\ )\mathbf{\Omega}_{\mathrm{R}}. \quad \text{(GN)} \tag{34}$$

We propose now the following objective corotational rate,

$$(\overset{\nabla}{\ }) = (\dot{\ }) - \mathbf{\Omega}(\ ) + (\ )\mathbf{\Omega}, \quad \text{(SS)}, \tag{35}$$

where $\mathbf{\Omega}$ is exactly defined by (12) and (22)–(25). The rate (35) corotational with respect to the substructure rotation $\mathbf{Q}$ and shortly denoted in this paper by SS-rate is connected with the GN-rate by eqns (32). In the next sections we will show that the SS-rate is appropriate for hyperelasticity as well as finite elastoplasticity and that it is identical with the GN-rate for the special cases of hypoelasticity and rigid-plasticity, while the ZJ-rate (33) yields a sufficient approximation for moderately large elastic and/or moderately large plastic strains.

4. APPROXIMATIONS FOR THE SPIN TENSORS

Finite elastoplasticity comprises important subclasses of material behaviour as rigid-plasticity and hypoelasticity. Therefore it is worthwhile to investigate the general results of the previous section for restricted elastic and/or plastic strains.

4.1 *Moderate elastic, finite plastic strains*

Let us consider first the case of moderate elastic, finite plastic strains defined by

$$\mathbf{V}^{\mathrm{e}} = \mathbf{1} + \boldsymbol{\varepsilon}^{\mathrm{e}}, \quad \boldsymbol{\varepsilon}^{\mathrm{e}} = \mathbf{O}(\theta), \quad \theta^2 \ll 1, \tag{36}$$

where $\mathbf{O}(\theta)$ denotes a tensor with eigenvalues of the order θ, and θ is a small number. Additionally we assume that the elastic strain velocity $\overset{\nabla}{\mathbf{V}}{}^{\mathrm{e}}$ is moderately large,

$$\overset{\nabla}{\mathbf{V}}{}^{\mathrm{e}} = \mathbf{O}(\theta), \tag{37}$$

while the plastic stretch $\mathbf{V}^{\mathrm{p}}$ and its rate may be finite,

$$\mathbf{V}^{\mathrm{p}} = \mathbf{1} + \boldsymbol{\varepsilon}^{\mathrm{p}}, \quad \boldsymbol{\varepsilon}^{\mathrm{p}} = \mathbf{O}(1), \quad \overset{\nabla}{\mathbf{V}}{}^{\mathrm{p}} = \mathbf{O}(1). \tag{38}$$

With (36)–(38) we can estimate the magnitude of all terms in eqns (20), (21) yielding

$$\mathbf{w}^{\mathrm{p}} = \mathbf{O}(1) \tag{39}$$

and

$$\begin{aligned} \mathbf{w}^{\mathrm{e}} &= \tfrac{1}{2}(\overset{\nabla}{\mathbf{V}}{}^{\mathrm{e}}(\mathbf{1} - \boldsymbol{\varepsilon}^{\mathrm{e}} + \mathbf{O}(\theta^2))) - (\mathbf{1} - \boldsymbol{\varepsilon}^{\mathrm{e}} + \mathbf{O}(\theta^2))\overset{\nabla}{\mathbf{V}}{}^{\mathrm{e}}) \\ &= \tfrac{1}{2}(\boldsymbol{\varepsilon}^{\mathrm{e}}\overset{\nabla}{\mathbf{V}}{}^{\mathrm{e}} - \overset{\nabla}{\mathbf{V}}{}^{\mathrm{e}}\boldsymbol{\varepsilon}^{\mathrm{e}}) + \mathbf{O}(\theta^2 \|\overset{\nabla}{\mathbf{V}}{}^{\mathrm{e}}\|) = \mathbf{O}(\theta^2). \end{aligned} \tag{40}$$

Thus the spin (22) can be approximated by

$$\mathbf{\Omega} = \mathbf{w} - \mathbf{w}^{\mathrm{p}} + \mathbf{O}(\theta^2), \tag{41}$$

with $\mathbf{w}^{\mathrm{p}}$ according to (25). The accuracy of this approximation is sufficient, because $\mathbf{\Omega}$ is needed only in the corotational rate formulae (14) and (35), respectively, where according to the application of $\boldsymbol{\Omega}$ the resulting error is of higher order small.

4.2 *Small elastic, finite plastic strains*

In classical metal alloys finite plastic strains occur regularly in combination with small elastic strains,

$$\mathbf{V}^{e} = \mathbf{1}+\boldsymbol{\varepsilon}^{e}, \quad \boldsymbol{\varepsilon}^{e} = \mathbf{O}(\theta^{2}). \tag{42}$$

We assume here also small elastic strain velocities

$$\overset{\nabla}{\mathbf{V}}{}^{e} = \mathbf{O}(\theta^{2}). \tag{43}$$

With (42), (43) the plastic spin (21) reduces to

$$\mathbf{w}^{p} = \tfrac{1}{2}(\overset{\nabla}{\mathbf{V}}{}^{p}\mathbf{V}^{p-1} - \mathbf{V}^{p-1}\overset{\nabla}{\mathbf{V}}{}^{p})(\mathbf{1}+\mathbf{O}(\theta^{2})) \tag{44}$$

and the spin (22) to

$$\boldsymbol{\Omega} = \mathbf{w}-\mathbf{w}^{p}. \tag{45}$$

Throughout the literature we can find the assumption that for small elastic – finite plastic deformations without kinematic hardening the plastic spin $\mathbf{w}^{p}$ vanishes. This is incorrect, because from (44) we can see that $\mathbf{w}^{p}$ vanishes only if $\overset{\nabla}{\mathbf{V}}{}^{p}$ and $\mathbf{V}^{p}$ are coaxial, which is in general not the case even without any hardening.

4.3 *Rigid-plastic deformations*

For rigid-plasticity the spin $\boldsymbol{\Omega}$ is exactly

$$\boldsymbol{\Omega} = \mathbf{w}-\mathbf{w}^{p} \tag{46}$$

with $\mathbf{w}^{p}$ according to (44) without the error term $\mathbf{O}(\theta^{2})$. From (27) to (30) it follows that $\overset{*}{\mathbf{R}} = \mathbf{1}$ and from (32) that the spin $\boldsymbol{\Omega}$ is identical with the spin $\boldsymbol{\Omega}_{R}$ and that the SS-rate (35) is identical with the GN-rate (34).

4.4 *Moderate elastic, moderate plastic strains*

In the case of moderately large elastic and moderately large plastic strains defined by the error estimations

$$\mathbf{V}^{e} = \mathbf{1}+\boldsymbol{\varepsilon}^{e}, \quad \boldsymbol{\varepsilon}^{e} = \mathbf{O}(\theta), \quad \overset{\nabla}{\mathbf{V}}{}^{e} = \mathbf{O}(\theta), \tag{47}$$

and

$$\mathbf{V}^{p} = \mathbf{1}+\boldsymbol{\varepsilon}^{p}, \quad \boldsymbol{\varepsilon}^{p} = \mathbf{O}(\theta), \quad \overset{\nabla}{\mathbf{V}}{}^{p} = \mathbf{O}(\theta), \tag{48}$$

we have $\mathbf{w}^{e} = \mathbf{O}(\theta^{2})$ according to (40) and

$$\mathbf{w}^{p} = \tfrac{1}{2}[\overset{\nabla}{\mathbf{V}}{}^{p}\mathbf{V}^{p-1}(\mathbf{1}+\mathbf{O}(\theta)) - \mathbf{V}^{p-1}\overset{\nabla}{\mathbf{V}}{}^{p}(\mathbf{1}+\mathbf{O}(\theta))] = \mathbf{O}(\theta^{2}), \tag{49}$$

and eqn (22) reduces to

$$\boldsymbol{\Omega} = \mathbf{w}+\mathbf{O}(\theta^{2}). \tag{50}$$

This means, for moderate elastic – moderate plastic strains the Zaremba–Jaumann rate can be used as sufficiently accurate corotational rate. The well-known oscillations for the simple shear problem occur only in the range of finite elastic and/or finite plastic strains. This will be shown in Section 7 also numerically.

4.5 *Finite elastic, small plastic strains*

Here we assume the following strain and strain rate estimations

$$\mathbf{V}^e = \mathbf{1}+\boldsymbol{\varepsilon}^e, \quad \boldsymbol{\varepsilon}^e = \mathbf{O}(1), \quad \overset{\nabla}{\mathbf{V}}{}^e = \mathbf{O}(1),$$
$$\mathbf{V}^p = \mathbf{1}+\boldsymbol{\varepsilon}^p, \quad \boldsymbol{\varepsilon}^p = \mathbf{O}(\theta^2), \quad \overset{\nabla}{\mathbf{V}}{}^p = \mathbf{O}(\theta^2). \tag{51}$$

Then $\mathbf{w}^e$ and $\mathbf{w}^p$ can be estimated as

$$\mathbf{w}^e = \mathbf{O}(1), \quad \mathbf{w}^p = \mathbf{O}(\theta^4) \tag{52}$$

leading to

$$\boldsymbol{\Omega} = \mathbf{w}-\mathbf{w}^e+\mathbf{O}(\theta^4), \tag{53}$$

where $\mathbf{w}^e$ is defined by (24).

4.6 *Hypoelasticity*

In hypoelasticity with vanishing plastic strains and strain rates the spin $\boldsymbol{\Omega}$ is given exactly by

$$\boldsymbol{\Omega} = \mathbf{w}-\mathbf{w}^e. \tag{54}$$

As in rigid-plasticity we obtain from (27) to (30) that $\overset{*}{\mathbf{R}} = \mathbf{1}$, and from (32) that $\boldsymbol{\Omega}$ is identical with $\boldsymbol{\Omega}_R$. Therefore the SS-rate is identical with the GN-rate. We can conclude from (54) that in hypoelasticity only the identical SS- and GN-rates can be used, but not the Zaremba–Jaumann rate (33), as is done in the literature.

5. COMPARISON WITH THE CONSTITUTIVE ASSUMPTIONS FOR THE PLASTIC SPIN PROPOSED IN THE LITERATURE

We want to compare, as far as it is possible, the exact plastic spin formula (25) with the constitutive assumptions for the plastic spin proposed and applied in the literature. Using the representation theorem Dafalias (1983) and Loret (1983) introduced the following functional form for the plastic spin [see also Van der Giessen (1991), eqn (9)],

$$\mathbf{w}^p = \langle\lambda\rangle[\eta_1(\boldsymbol{\alpha\sigma}-\boldsymbol{\sigma\alpha})+\eta_2(\boldsymbol{\alpha}^2\boldsymbol{\sigma}-\boldsymbol{\sigma\alpha}^2)+\eta_3(\boldsymbol{\alpha\sigma}^2-\boldsymbol{\sigma}^2\boldsymbol{\alpha})$$
$$+\eta_4(\boldsymbol{\alpha\sigma\alpha}^2-\boldsymbol{\alpha}^2\boldsymbol{\sigma\alpha})+\eta_5(\boldsymbol{\sigma\alpha\sigma}^2-\boldsymbol{\sigma}^2\boldsymbol{\alpha\sigma})], \tag{55}$$

where $\langle\ \rangle$ are the Macauly brackets, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{\alpha}$ the internal structure variable (backstress tensor) and η_i $(i = 1,\ldots,5)$ unknown scalar valued functions of the invariants tr $\boldsymbol{\sigma}$, tr $\boldsymbol{\sigma}^2$, tr $\boldsymbol{\sigma}^3$, tr $\boldsymbol{\alpha}$, tr $\boldsymbol{\alpha}^2$, tr $\boldsymbol{\alpha}^3$, tr$(\boldsymbol{\alpha\sigma})$, tr$(\boldsymbol{\alpha}^2\boldsymbol{\sigma})$, tr$(\boldsymbol{\alpha\sigma}^2)$, tr$(\boldsymbol{\alpha}^2\boldsymbol{\sigma}^2)$.

Assuming a simple flow rule with the plastic strain rate $\mathbf{d}^p$ proportional to $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}-\boldsymbol{\alpha}$, respectively, the lowest order approximation of (55) used in the literature [e.g. Loret (1983), Onat (1984), Dafalias (1985a,b), Van der Giessen (1991, eqn 10)] is

$$\mathbf{w}^p = \frac{\rho}{2\zeta}(\boldsymbol{\alpha}\mathbf{d}^p-\mathbf{d}^p\boldsymbol{\alpha}), \tag{56}$$

where in order to obtain (56) in a dimensionless form we introduced the additional coefficient ζ, the kinematic hardening modulus according to (71) and (75), respectively.

In the literature formula (56) is applied to rigid-plastic and small elastic – finite plastic deformations with kinematic hardening. Paulun and Pecherski (1992) determined numerically the unknown coefficient ρ for the simple shear problem.

Comparing the exact plastic spin (25) with the constitutive assumption (55) with the unknown coefficients η_i, which still have to be determined, it is obvious that formula (25) is not only exact, but also easy to implement into a general solution algorithm.

Next we want to compare a simplest first order approximation of the exact formula (21) and (25), respectively, with (56). For small elastic – finite plastic strains eqn (21) reduces to (44). Furthermore for extended moderately large plastic strains ($\|\boldsymbol{\varepsilon}^p\| \leqslant 0(\sqrt[3]{\theta^2})$) the corotational plastic strain rate $\overset{\nabla}{\mathbf{V}}{}^p$ in eqn (44) can sufficiently be approximated by

$$\overset{\nabla}{\mathbf{V}}{}^p \approx \mathbf{d}^p \tag{57}$$

and also the backstress tensor $\boldsymbol{\alpha}$ according to (75) by

$$\boldsymbol{\alpha} \approx \zeta(\mathbf{1} - \mathbf{V}^{p-1}). \tag{58}$$

Introducing (57) and (58) into (44) leads to the following first-order approximation of the plastic spin with simplest kinematic hardening rule,

$$\mathbf{w}^p \approx \frac{1}{2\zeta}(\boldsymbol{\alpha}\mathbf{d}^p - \mathbf{d}^p\boldsymbol{\alpha}). \tag{59}$$

A comparison of (56) and (59) shows that the constitutive assumption (56) of Dafalias is a good approximation of the exact plastic spin for small elastic and extended moderately large plastic strains with kinematic hardening provided ρ is chosen to be one. This is confirmed also numerically and shown in Fig. 4. It is also seen that it is incorrect to conclude from (56) that for vanishing backstress tensor, $\boldsymbol{\alpha} \to \mathbf{0}$, also the plastic spin vanishes, $\mathbf{w}^p \to \mathbf{0}$. For $\boldsymbol{\alpha} \to \mathbf{0}$ also $\zeta \to 0$ and the plastic spin $\mathbf{w}^p$ is given by formula (44) for small elastic strains.

For various material behaviour we analyse in Section 7 the plastic spin numerically for the simple shear problem and compare it with the above mentioned results of the literature.

6. CONSTITUTIVE EQUATIONS

Using the proposed corotational SS-rate (35) with (22)–(25) we formulate in this section the elastic–plastic constitutive equations for moderately large elastic and finite plastic strains with isotropic and kinematic hardening. We assume that the elastic material behavior is isotropic and that the influence of texturing can be neglected.

6.1 *Elastic constitutive equations*

Anand (1979, 1986) showed that for moderately large elastic strains the elastic potential W referred to the initial volume is a quadratic function of the logarithmic elastic strain tensor. Correspondingly, for finite elastoplasticity we assume the existence of an elastic potential of the form

$$W = \tfrac{1}{2}\ln \mathring{\mathbf{U}}^e \cdot \mathbb{C}^e \cdot \ln \mathring{\mathbf{U}}^e, \tag{60}$$

where $\mathbb{C}^e$ is the known constant fourth-order isotropic elasticity tensor of the linear theory, and a dot denotes the tensor product with contraction of two indices.

For isotropic finite elastic deformations the Lagrangean logarithmic strain tensor $\ln \mathbf{U}$ is power-conjugate [in the sense of Hill (1968)] to the back-rotated Kirchhoff stress tensor $\mathbf{R}^T\boldsymbol{\tau}\mathbf{R}$ [see also Wang and Truesdell (1977) and Hoger (1987)]. Applying this result to finite elasto-plasticity with the basic unique decomposition of the deformation gradient,

$\mathbf{F} = \mathbf{Q}\mathring{\mathbf{U}}^e\mathbf{U}^p$, the back-rotated Kirchhoff stress tensor $\mathring{\boldsymbol{\tau}}$, referred to the initial volume, is power-conjugate to the back-rotated logarithmic elastic strain tensor $\ln \mathring{\mathbf{U}}^e$. Therefore we obtain from (60)

$$\mathring{\boldsymbol{\tau}} = \frac{\partial W}{\partial(\ln \mathring{\mathbf{U}}^e)} = \mathbb{C}^e \cdot \ln \mathring{\mathbf{U}}^e. \tag{61}$$

Additional investigations concerning the work-conjugacy between $\mathring{\boldsymbol{\tau}}$ and $\ln \mathring{\mathbf{U}}^e$ can be found in Appendix B.

The rate form of eqn (61) yields

$$\mathring{\dot{\boldsymbol{\tau}}} = \mathbb{C}^e \cdot (\ln \mathring{\mathbf{U}}^e)^{\cdot}. \tag{62}$$

With the back-rotated elastic deformation rate $\mathring{\mathbf{d}}^e$,

$$\mathring{\mathbf{d}}^e = \mathbf{Q}^T\mathbf{d}^e\mathbf{Q} = \tfrac{1}{2}(\dot{\mathring{\mathbf{U}}}^e\mathring{\mathbf{U}}^{e-1} + \mathring{\mathbf{U}}^{e-1}\dot{\mathring{\mathbf{U}}}^e), \tag{63}$$

and the operator $\partial(\ln \mathring{\mathbf{U}}^e)^{\cdot}/\partial\mathring{\mathbf{d}}^e$ (see Appendix B) eqn (62) can be transformed to

$$\mathring{\dot{\boldsymbol{\tau}}} = \mathbb{C}^e \cdot \frac{\partial(\ln \mathring{\mathbf{U}}^e)^{\cdot}}{\partial\mathring{\mathbf{d}}^e} \cdot \mathring{\mathbf{d}}^e = \hat{\mathbb{C}}^e(\mathring{\mathbf{U}}^e) \cdot \mathring{\mathbf{d}}^e, \tag{64}$$

where

$$\hat{\mathbb{C}}^e(\mathring{\mathbf{U}}^e) := \mathbb{C}^e \cdot \frac{\partial(\ln \mathring{\mathbf{U}}^e)^{\cdot}}{\partial\mathring{\mathbf{d}}^e} \tag{65}$$

has the same symmetry properties as $\mathbb{C}^e$. This can be shown by investigating the properties of the operator $\partial(\ln \mathring{\mathbf{U}}^e)^{\cdot}/\partial\mathring{\mathbf{d}}^e$ (see Appendix B).

Push-forward of eqns (61) and (64) with $\mathbf{Q}$ leads to the elastic constitutive equation referred to the actual configuration with the Kirchhoff stress tensor $\boldsymbol{\tau}$,

$$\boldsymbol{\tau} = \mathbf{Q}\mathring{\boldsymbol{\tau}}\mathbf{Q}^T = \mathbb{C}^e \cdot \ln \mathbf{V}^e \tag{66}$$

and its corotational SS-rate

$$\overset{\nabla}{\boldsymbol{\tau}} = \mathbf{Q}\mathring{\dot{\boldsymbol{\tau}}}\mathbf{Q}^T = \dot{\boldsymbol{\tau}} - \boldsymbol{\Omega}\boldsymbol{\tau} + \boldsymbol{\tau}\boldsymbol{\Omega} = \hat{\mathbb{C}}^e(\mathbf{V}^e) \cdot \mathbf{d}^e. \tag{67}$$

In (67) $\hat{\mathbb{C}}^e(\mathbf{V}^e)$ is the push-forward of $\hat{\mathbb{C}}^e(\mathring{\mathbf{U}}^e)$ and can be determined analogously to eqn (65) replacing $\mathring{\mathbf{U}}^e$ by $\mathbf{V}^e$ and $\mathring{\mathbf{d}}^e$ by $\mathbf{d}^e$.

We want to point out that the application of the SS-rate (35) in the elastic rate constitutive eqn (67) is derived in a straightforward way. It does not depend on the chosen elastic strain measure or on the chosen elastic potential W, thus eqn (67) is also valid for arbitrarily large elastic strains.

Within the moderate strain assumption one can show that the following estimation is valid,

$$(\ln \mathring{\mathbf{U}}^e)^{\cdot} = \mathring{\mathbf{d}}^e(\mathbf{1} + \mathbf{O}\|\ln \mathring{\mathbf{U}}^e\|^2), \tag{68}$$

leading to the approximated elastic rate constitutive equation for moderately large elastic strains

$$\overset{\vee}{\tau} = \mathbb{C}^e \cdot \mathbf{d}^e(\mathbf{1} + \mathbf{O}(\|\ln \mathbf{V}^e\|^2)). \tag{69}$$

In order to eliminate small accumulation errors in the course of a step-by-step calculation the actual stress should be recalculated from time to time using eqn (66).

6.2 *Backstress tensor*

Analogously to the notions of hyper- and hypoelasticity we can formulate the constitutive equations of the backstress tensor for hyper- or hypoplasticity. In the first case we assume the existence of a potential of the backstress tensor, in the second case there is no potential of the backstress tensor, which is the usual assumption in finite elastoplasticity.

Taking into account the microstructural behaviour of continua [see Le and Stumpf (1994), (1995a)], we may assume the existence of a potential ψ for the backstress tensor $\boldsymbol{\alpha}$,

$$\psi = \psi(\ln \mathbf{U}^p). \tag{70}$$

Analogous to the elastic constitutive eqn (61) the simplest functional form for $\mathring{\boldsymbol{\alpha}}$ is then

$$\mathring{\boldsymbol{\alpha}} = \zeta \ln \mathbf{U}^p, \tag{71}$$

where ζ is the kinematic hardening modulus. The rate of (71) is obtained as

$$\dot{\mathring{\boldsymbol{\alpha}}} = \zeta(\ln \mathbf{U}^p)^{\cdot}. \tag{72}$$

With the back-rotated plastic strain rate $\mathring{\mathbf{d}}^p$,

$$\mathring{\mathbf{d}}^p = \mathbf{Q}^T \mathbf{d}^p \mathbf{Q}, \tag{73}$$

and the operator $\partial(\ln \mathbf{U}^p)/\partial \mathbf{U}^p$, which is derived in Appendix C, (72) can be transformed to

$$\dot{\mathring{\boldsymbol{\alpha}}} = \zeta \frac{\partial(\ln \mathbf{U}^p)}{\partial \mathbf{U}^p} \cdot \frac{\partial \dot{\mathbf{U}}^p}{\partial \mathring{\mathbf{d}}^p} \cdot \mathring{\mathbf{d}}^p. \tag{74}$$

Push-forward of eqns (71) and (74) with $\mathbf{Q}$ yields the constitutive equation for the backstress in the actual configuration

$$\boldsymbol{\alpha} = \mathbf{Q}\mathring{\boldsymbol{\alpha}}\mathbf{Q}^T = \zeta \ln \mathbf{V}^p \tag{75}$$

and its corotational SS-rate

$$\begin{aligned} \overset{\vee}{\boldsymbol{\alpha}} = \mathbf{Q}\dot{\mathring{\boldsymbol{\alpha}}}\mathbf{Q}^T &= \dot{\boldsymbol{\alpha}} - \boldsymbol{\Omega}\boldsymbol{\alpha} + \boldsymbol{\alpha}\boldsymbol{\Omega} \\ &= \zeta \frac{\partial(\ln \mathbf{V}^p)}{\partial \mathbf{V}^p} \cdot \mathbb{A}^{p-1} \cdot \mathbf{d}^p. \end{aligned} \tag{76}$$

In (76)

$$\mathbb{A}^{p-1} = \frac{\partial \overset{\vee}{\mathbf{V}}{}^p}{\partial \mathbf{d}^p} = \mathbf{Q} \frac{\partial \dot{\mathbf{U}}^p}{\partial \mathring{\mathbf{d}}^p} \mathbf{Q}^T \tag{77}$$

is used, which follows from (23) and eqns (A4), (A5) in Appendix A. The operator $\partial(\ln \mathbf{V}^p)/\partial \mathbf{V}^p$ can be calculated analogously to the considerations of Appendix C, replacing there $\mathbf{U}^p$ and $\dot{\mathbf{U}}^p$ by $\mathbf{V}^p$ and $\overset{\nabla}{\mathbf{V}}{}^p$, respectively. The application of the SS-rate (35) in the rate constitutive eqn (76) is mathematically stringent.

In the following sections we call constitutive equations for the backstress tensor of the type (75) or its associated rate form (76) "hyperplastic". By this we want to distinguish them from the rate constitutive equation of the form,

$$\overset{\nabla}{\boldsymbol{\alpha}} = \zeta \mathbf{d}^p, \tag{78}$$

which we call "hypoplastic". Within an error estimation one can show that for moderately large plastic and elastic strains (76) reduces to (78).

6.3 *Elastic–plastic tangential operator*

With the Kirchhoff stress tensor $\boldsymbol{\tau}$, the backstress tensor $\boldsymbol{\alpha}$, the average yield stress τ_y and a tensor-valued variable $\boldsymbol{\chi} = \boldsymbol{\chi}(\ln \mathbf{V}^p)$ describing the texturing, the yield condition can be formulated in the following form

$$F = F(\boldsymbol{\tau}, \boldsymbol{\alpha}, \tau_y, \boldsymbol{\chi}) \begin{cases} <0: & \text{no yielding} \\ =0: & \text{yielding.} \end{cases} \tag{79}$$

The elastic and plastic constitutive equations derived above result in the elastic–plastic constitutive equation

$$\boldsymbol{\tau} = \mathbb{C}^{ep} \cdot \mathbf{d}, \tag{80}$$

where $\mathbf{d}$ is the total strain rate (16) and $\mathbb{C}^{ep}$ the elastic–plastic tangential operator

$$\mathbb{C}^{ep} = \hat{\mathbb{C}}^e - \frac{(\hat{\mathbb{C}}^e \cdot F_{,\tau}) \otimes (F_{,\tau} \cdot \hat{\mathbb{C}}^e)}{F_{,\tau} \cdot \hat{\mathbb{C}}^e \cdot F_{,\tau} - F_{,\ln \mathbf{V}^p} \cdot \dfrac{\partial (\ln \mathbf{V}^p)}{\partial \mathbf{V}^p} \cdot \mathbb{A}^{p-1} \cdot F_{,\tau}} \tag{81}$$

with

$$F_{,\ln \mathbf{V}^p} = F_{,\tau_y} \frac{\partial \tau_y}{\partial (\ln \mathbf{V}^p)} + \zeta F_{,\alpha} + F_{,\chi} \cdot \frac{\partial \boldsymbol{\chi}}{\partial (\ln \mathbf{V}^p)}. \tag{82}$$

7. NUMERICAL APPLICATIONS

In this section we want to apply our general concept to analyse numerically the simple shear problem for various material behaviour ranging from rigid-plastic to finite elastic–finite plastic up to hyperelastic behaviour and to compare the results with those of the literature. For simplicity we consider idealized materials with "linear" hardening laws, and we do not take into account the evolution of elastic anisotropies due to texturing and its influence on the deformation of the yield surface. We use the von Mises yield condition with associated flow rule.

Also in this section we consider a closed cyclic finite elastic deformation for hyper- and hypoelastic material. It will be shown, that only for hyperelastic and "associated hypoelastic" materials and only by application of the SS-rate (35) and the GN-rate (34), which are identical in this case, a closed deformation cycle leads to vanishing residual

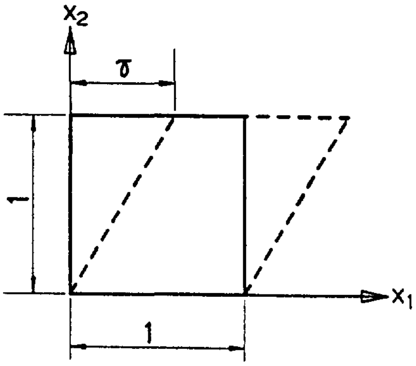

Fig. 1. Simple shear.

stresses. By "associated hypoelastic" we mean that an elastic potential exists. The ZJ-rate (33) leads to residual stresses after a closed cycle [see also Kojic and Bathe (1987)] and is not an appropriate corotational rate for finite strains, as was shown analytically in the previous sections.

7.1 *Rigid-plastic material under simple shear*

The geometry of the simple shear problem is sketched in Fig. 1. This problem is analysed first for rigid-plastic material with linear isotropic hardening,

$$\dot{\tau}_y = h_i \sqrt{\tfrac{2}{3}} \, \|\mathbf{d}^p\|, \tag{83}$$

where h_i is the isotropic hardening modulus and τ_y the average yield stress. We assume either "hyperplastic" kinematic hardening according to Section 6.2,

$$\boldsymbol{\alpha} = \zeta \ln \mathbf{V}^p, \tag{84}$$

or "hypoplastic" hardening,

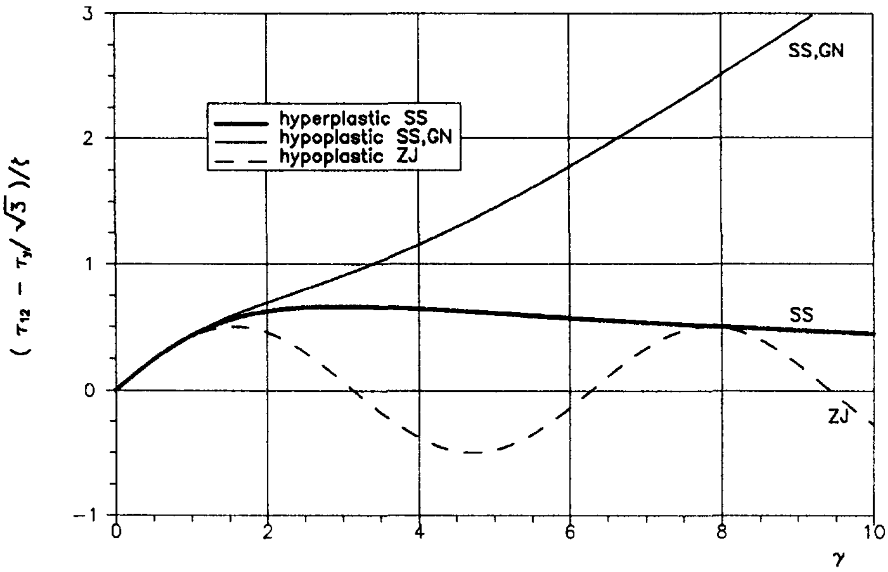

Fig. 2a. Simple shear; rigid-plastic material: dimensionless shear stress.

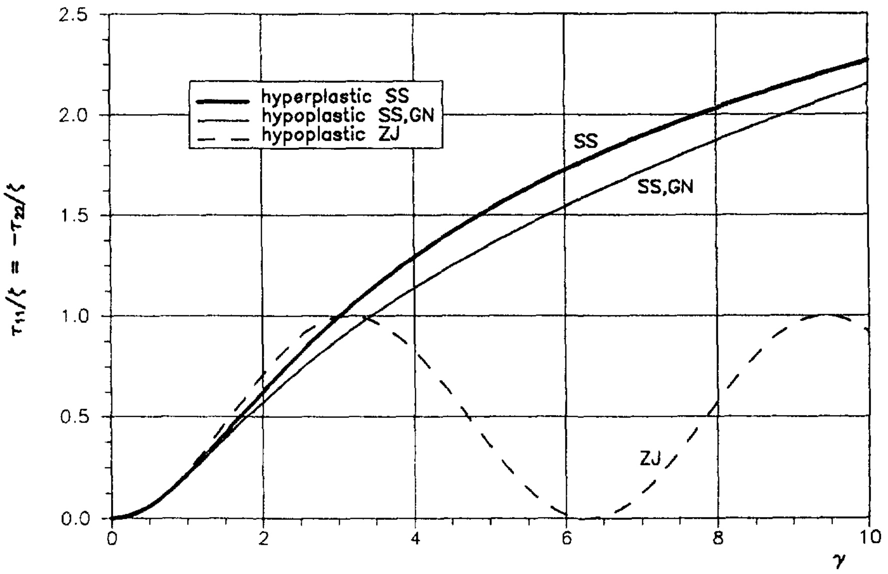

Fig. 2b. Simple shear; rigid-plastic material: dimensionless normal stresses.

$$\overset{\nabla}{\alpha} = \zeta \mathbf{d}^{p}, \tag{85}$$

where ζ is the kinematic hardening modulus. The results are shown in Figs 2(a), 2(b) and 3(a), 3(b) for the stresses and in Fig. 4 for the plastic spin. Due to the dimension-free presentation no explicit constitutive parameters are needed here. Therefore, the results are valid also in the limiting case $\zeta \to 0$, where no kinematic hardening occurs. It can be seen that also in this case the plastic spin does not vanish.

The stress response of the "hyperplastic" material follows the logarithmic plastic strain tensor, which produces a maximum dimensionless shear stress at $\gamma \approx 3$, while the normal stresses are increasing monotonously. For the "hypoplastic" material various rates are examined and compared, the SS-rate (36), the GN-rate (34), the Dafalias rate D according to (56) for $\rho = 1.0, 0.5, 0.3$ and the ZJ-rate (33). One can see that up to $\gamma \approx 1.8$ the Dafalias rate D1.0 for $\rho = 1.0$ produces nearly identical results as the SS- and the GN-rate confirming our theoretical considerations of Section 5. Differences are observed for $\rho = 0.5$, D0.5, and $\rho = 0.3$, D0.3, where for the latter one oscillations occur similar to the ZJ-rate.

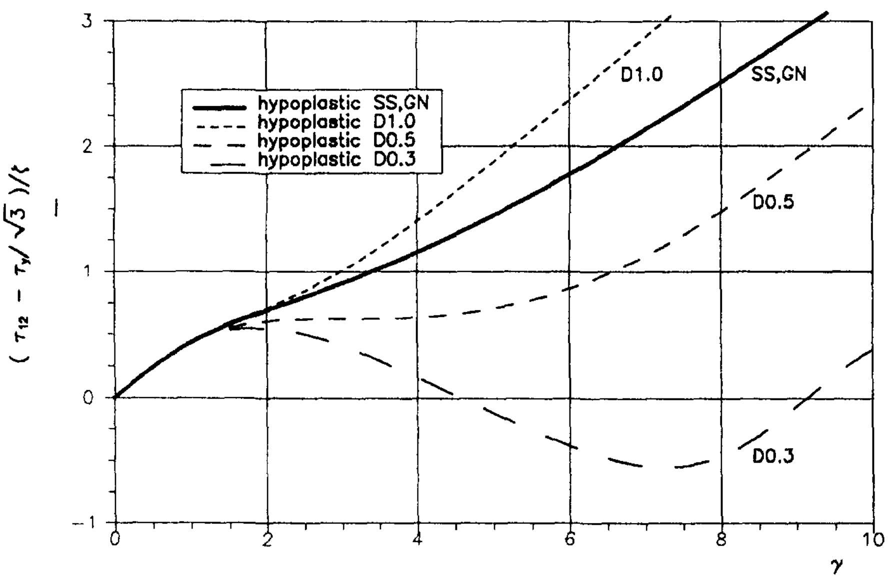

Fig. 3a. Simple shear: rigid-plastic material: dimensionless shear stress.

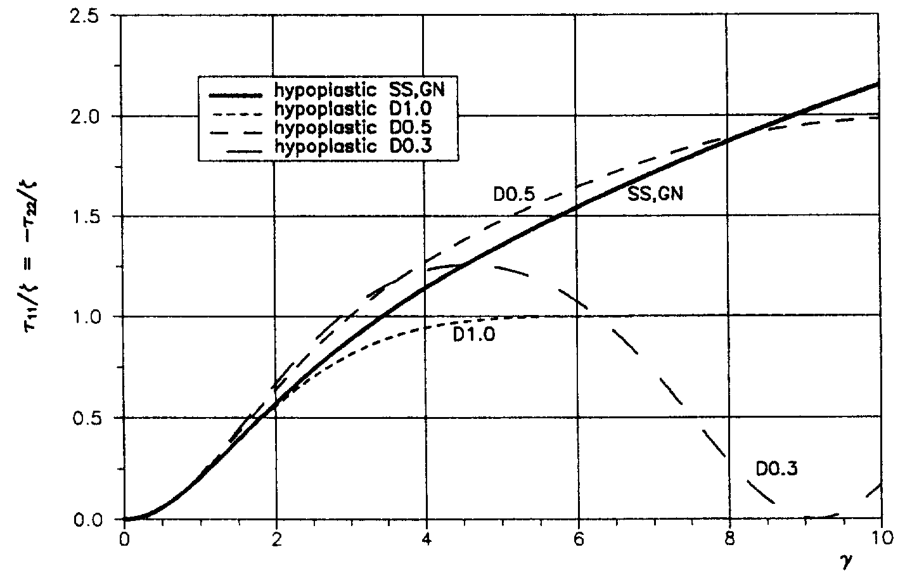

Fig. 3b. Simple shear; rigid-plastic material: dimensionless normal stresses.

7.2 Finite elastic material behaviour under a closed deformation cycle

In Kojic and Bathe (1987) a rectangular deformation cycle is analysed for a hypoelastic material by using the ZJ-rate. After performing a closed cycle, residual stresses were obtained. In this section we investigate a circular deformation cycle with the deformation gradient **F** given in Cartesian coordinates as

$$[F_{\alpha\beta}] = \begin{bmatrix} 1 & 3\sin\varphi \\ 0 & 4-3\cos\varphi \end{bmatrix}, \quad \alpha,\beta \in \{1,2\}, \quad 0 \leqslant \varphi \leqslant 2\pi. \tag{86}$$

Figures 5(a)–(c) show that only the hyperelastic material, eqn (66), and the "associated hypoelastic" material, eqn (67), by application of the SS-rate or the GN-rate, which are

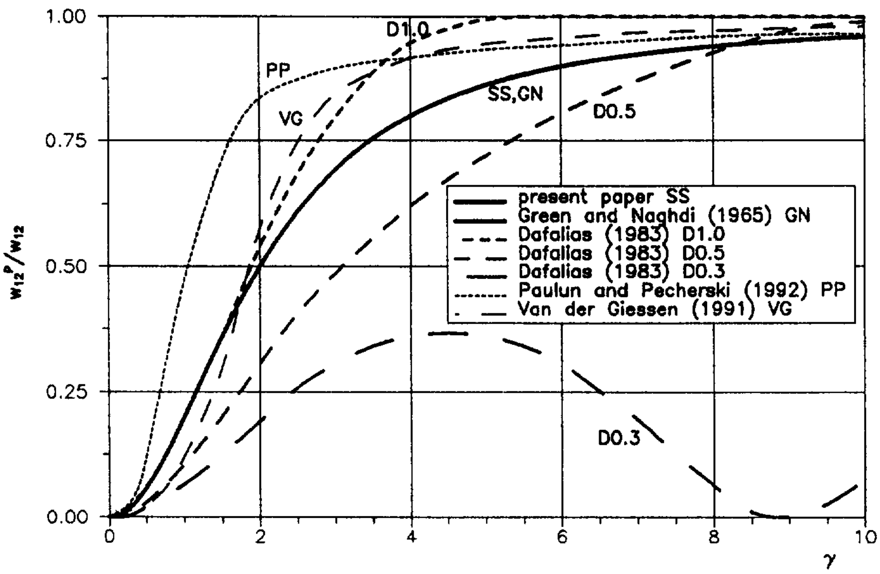

Fig. 4. Simple shear; rigid-plastic material: plastic spin.

identical in the present case, produce reasonable results with no residual stresses after a closed deformation cycle. The same can be shown for any elastic material, for which a potential exists, provided the SS- or GN-rate is used. All other rates lead to residual stresses, which is physically not acceptable.

7.3 *Moderate elastic, finite plastic material behaviour*

Now we consider a material which allows moderately large elastic and finite plastic strains undergoing simple shear, where

$$E = 20\tau_{yo}, \quad \nu = 0.3, \quad h_i = \sqrt{\frac{3}{2}}\frac{\partial \bar{t}_y}{\partial \|\mathbf{d}^p\|} = 0.2\tau_{yo}, \quad \zeta = 0.2\tau_{yo} \tag{87}$$

are chosen. Here E is Young's modulus, ν Poisson's ratio, τ_{yo} the initial yield stress, h_i the

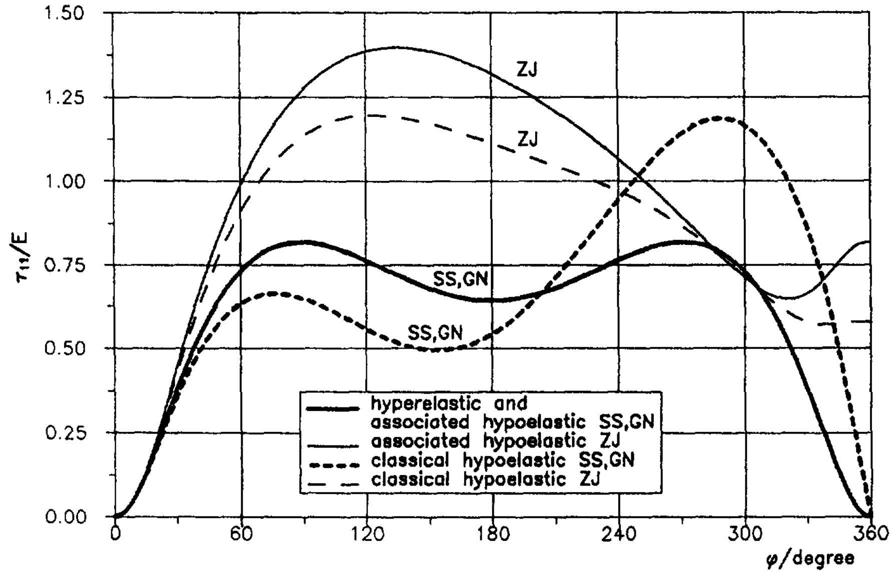

Fig. 5a. Finite elastic material under closed deformation cycle: ε_{11}/E.

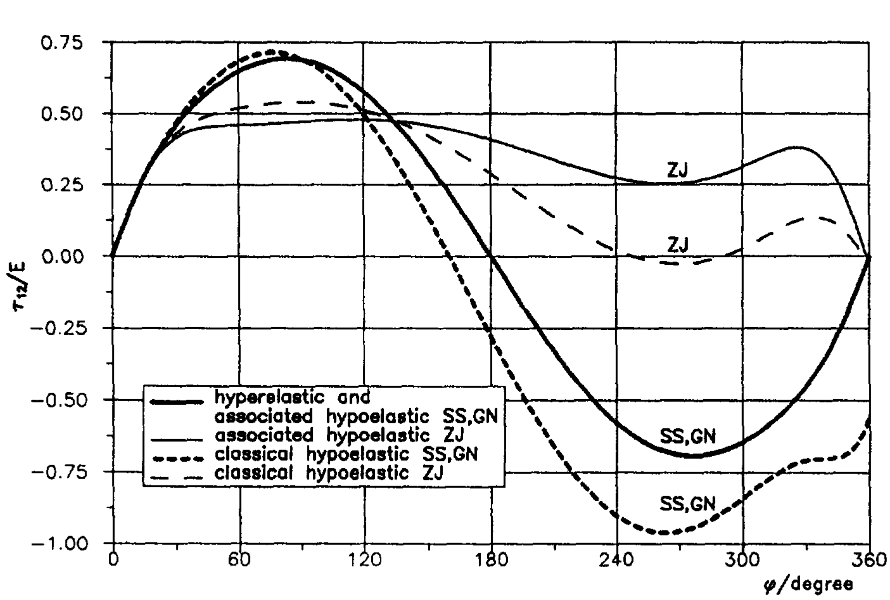

Fig. 5b. Finite elastic material under closed deformation cycle: τ_{12}/E.

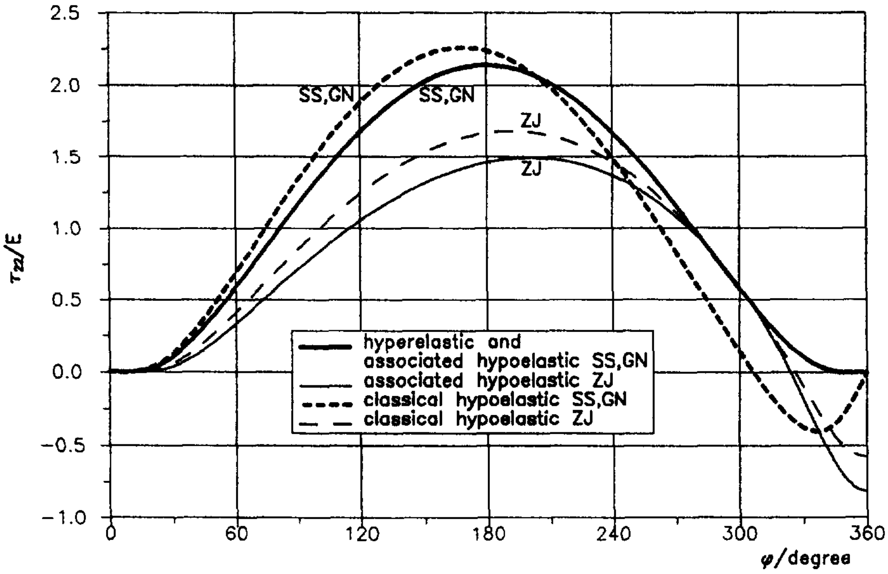

Fig. 5c. Finite elastic material under closed deformation cycle: τ_{22}/E.

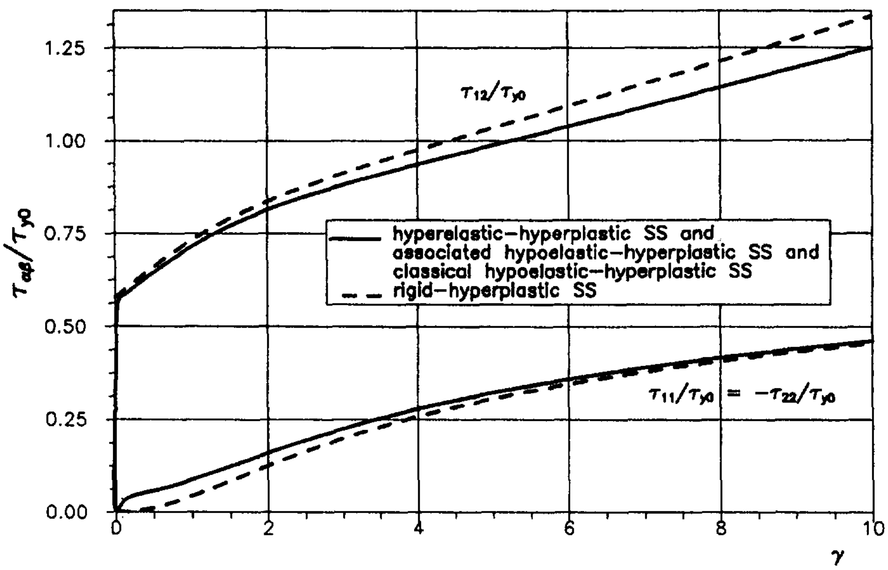

Fig. 6. Simple shear with moderate elastic, finite plastic strains.

isotropic hardening parameter and ζ the kinematic hardening modulus due to eqn (84) [or equivalently (76)], alternatively (85).

Figure 6 shows the results for the "hyperplastic" material according to eqns (84) and (76). Since the elastic strains are only moderately large, classical hypoelastic material description performs the same results as hyperelastic or "associated hypoelastic" descriptions, if the SS-rate is applied. These solutions are compared with those of a rigid-"hyperplastic" material using the SS-rate. One can see, that the differences between the solutions with reference to the actual yield stress are of the order of the elastic strains, which are increasing from about 6% to about 12% at $\gamma = 10$.

Since a "hyperplastic" material behaviour can only be described by applying the SS-rate, or mathematical inconsistency will occur, we choose the "hypoplastic" hardening according to eqn (85) in order to examine the influence of other rates. The results are

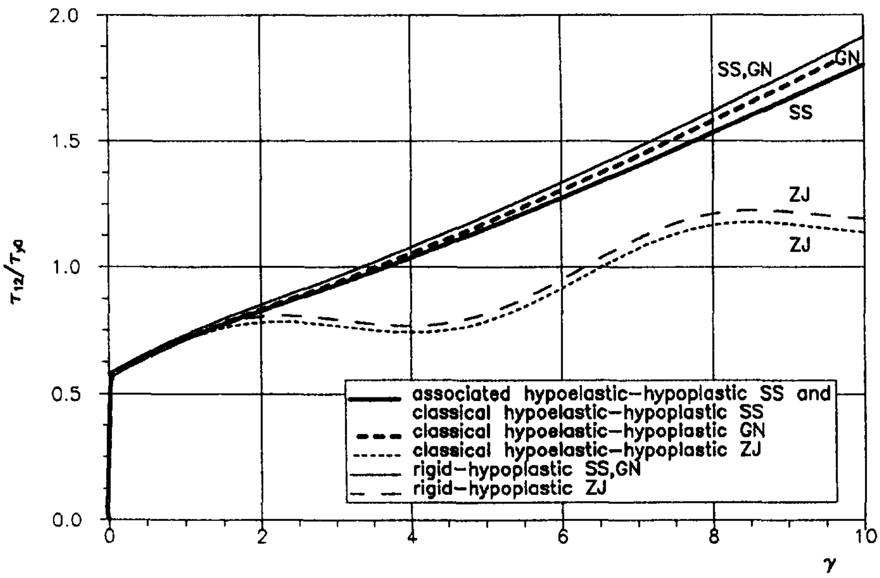

Fig. 7a. Simple shear with moderate elastic, finite plastic strains: shear stress.

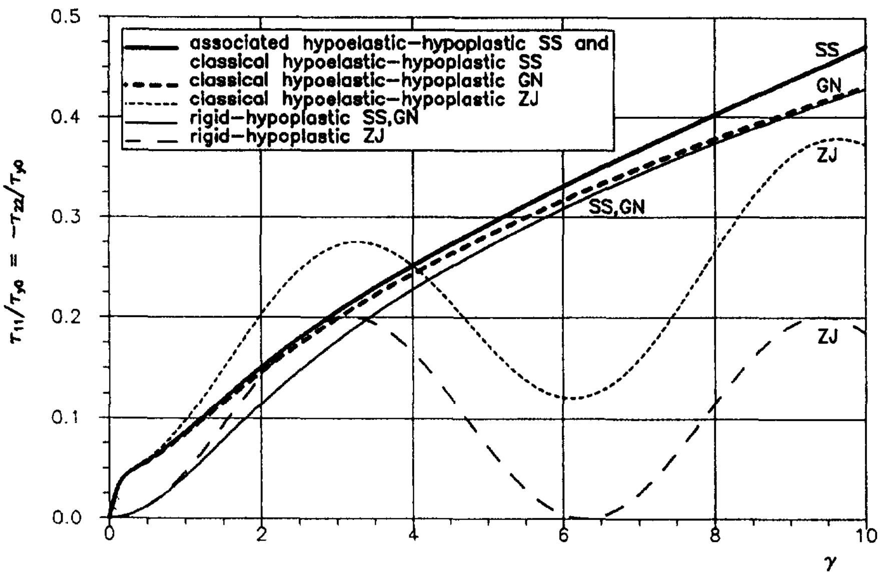

Fig. 7b. Simple shear with moderate elastic, finite plastic strains: normal stresses.

presented in Fig. 7(a) and (b). Again "associated" and classical hyperelastic materials lead to identical solutions, if the SS-rate is used. The difference to the GN-rate begins to be observable at $\gamma = 1$ and is increasing with γ. The results are limited by the rigid-"hypoplastic" solutions using the GN- and SS-rate. This can be expected, because the GN-rate does not account for the elastic deformation. The ZJ-rate leads always to the known oscillations.

7.4 *Finite elastic, finite plastic material behaviour without hardening*

Figures 8(a) and (b) show the response of an ideal-plastic material with Young's modulus $E = 2\tau_y/\sqrt{3}$, τ_y the average yield stress, Poisson's ratio $\nu = 0.3$, undergoing finite simple shear. We assume that there is no hardening. The elastic material response is hyperelastic according to eqn (66) or "associated hypoelastic" according to eqn (67) using

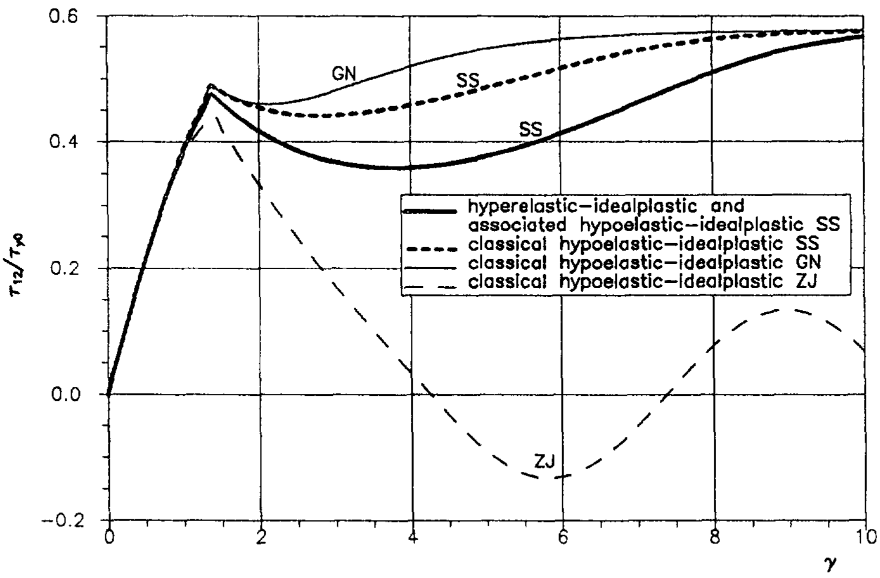

Fig. 8a. Simple shear with finite elastic, finite plastic strains, no hardening: shear stress.

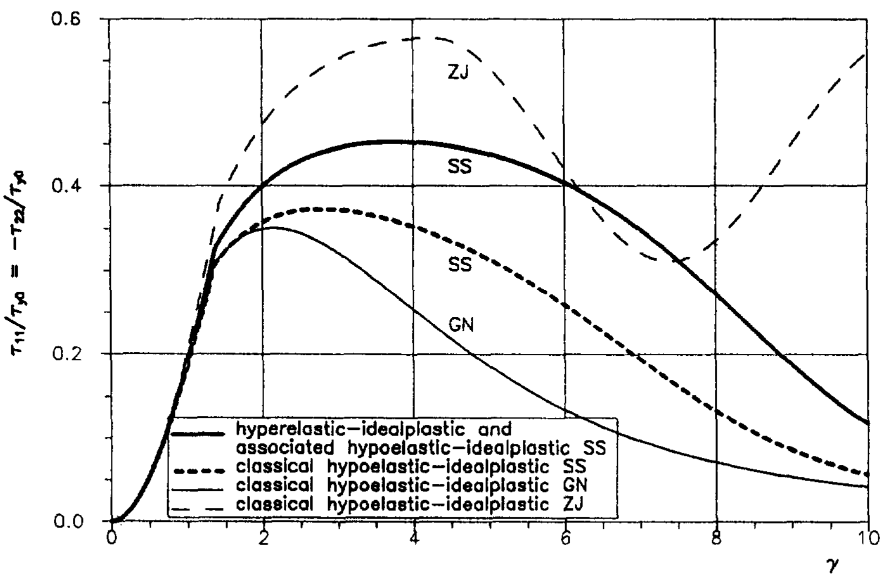

Fig. 8b. Simple shear with finite elastic, finite plastic strains, no hardening: normal stresses.

the SS-rate. Both variants lead to identical results. The "associated hypoelastic" material law can only be applied by using the SS-rate, otherwise mathematical inconsistency occurs.

Using the classical hypoelastic material law we can examine the influence of other rates. After a moderately large deformation step beyond the point of first yielding at $\gamma = 1.4$ the result due to the GN-rate diverges from that of the SS-rate. For very large values of γ the plastic strains become dominant compared with the elastic strains, because no hardening is assumed. Therefore, for large γ the deformation can be considered as approximately rigid-plastic with $\tau_{12} = \tau_y/\sqrt{3}$ and $\tau_{11} = \tau_{22} = 0$.

For the simple shear problem we have shown that the choice of the corotational rate is the crucial point in finite strain analysis. Especially, if there is no hardening the plastic spin $\mathbf{w}^p$ becomes the dominant contribution to the difference between the material spin $\mathbf{w}$ and the substructure spin $\mathbf{\Omega}$ according to (22)–(25), as can be seen in Fig. 9. Therefore, the assumption of many authors that the plastic spin is zero, if there is no kinematic hardening, is incorrect and leads to unacceptable results for finite elastic plastic deformations.

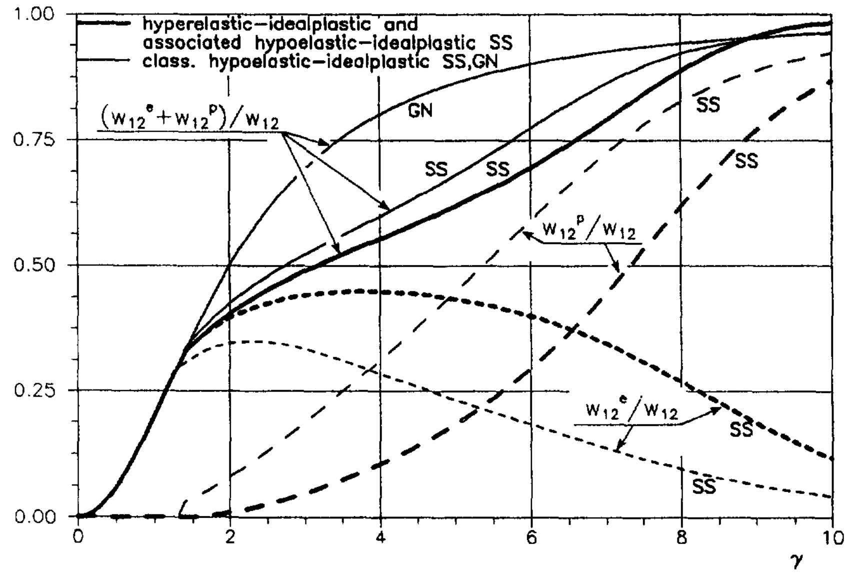

Fig. 9. Simple shear with finite elastic, finite plastic strains, no hardening: elastic and plastic spin.

7.5 *Unconstrained shear*

Lee and Wertheimer (1983), Paulun and Pecherski (1985) and Xia and Ellyin (1993) consider the so-called unconstrained shear problem with a deformation of the unit cube of Fig. 1 subject to prescribed shear stresses. The assumed material behaviour is hypoelastic and rigid-plastic. Comparative numerical results are presented in Xia and Ellyin (1993). Since for hypoelastic and rigid-plastic materials our corotational rate is identical with the Green–Naghdi rate, our numerical results are also identical with those obtained by Xia and Ellyin for the Green–Naghdi rate. In their paper Xia and Ellyin emphasize that their corotational rate furnishes a limiting value for $\tan \beta$ (β is the shear angle, see Fig. 1) when τ becomes large, a behaviour which is not predicted by any other corotational rate.

8. CONCLUSION

An objective kinematical and constitutive model for finite elastoplasticity is proposed. The presented exact formulae for elastic-, plastic- and substructure spin referred to the current configuration enable the formulation of a corotational rate appropriate for the whole range of finite elastoplasticity including hypoelasticity. This corotational rate is used to define appropriate constitutive and evolution equations for isotropic–elastic material with induced plastic flow undergoing isotropic and kinematic hardening and to derive the elastic–plastic tangential operator. The results presented in this paper can be easily implemented into general solution algorithms allowing one also to analyse hypoelastic deformations with vanishing residual stresses after closed deformation cycles. Detailed numerical applications to the simple shear problem and comparisons with results of the literature show the generality and applicability of the concept.

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APPENDIX

A. DETERMINATION OF THE FOURTH-ORDER TENSORS $\mathbb{A}^e$, $\mathbb{A}^p$

To determine the fourth-order tensors $\mathbb{A}^e$, $\mathbb{A}^p$ given by eqns (17), (18) and (23) let us introduce convective coordinates with covariant and contravariant base vector $\mathbf{g}_i$, $\mathbf{g}^i$ of the actual deformed configuration and with Latin indices running from 1 to 3. Using the common summation convention the tensor $\mathbb{A}^e$ can be written as:

$$\mathbb{A}^e = A^{e\,kl}_{ij}\,\mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}_k \otimes \mathbf{g}_l, \tag{A1}$$

where $\otimes$ denotes the usual tensor product.

With the covariant and contravariant metric tensors $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$, $g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j$ and with the Kronecker Symbol δ^j_i we obtain from (17) and (23) the components of $\mathbb{A}^e$ in the form

$$A^{e\,kl}_{ij} = \tfrac{1}{2}(\delta^k_i g_{jm}(\mathbf{V}^{e-1})^{ml} + \delta^l_i g_{jm}(\mathbf{V}^{e-1})^{mk}). \tag{A2}$$

The inverse operator $\mathbb{A}^{e-1}$ is obtained as solution of the linear algebraic equations

$$A^{e\,kl}_{ij}(\mathbb{A}^{e-1})^{mn}_{kl} = \delta^m_i \delta^n_j. \tag{A3}$$

The components $A^{p\,kl}_{ij}$ of the tensor $\mathbb{A}^p$ defined by (18) and (23) can be determined as

$$A^{p\,kl}_{ij} = \tfrac{1}{2}((\mathbf{V}^e)_{im} g^{mk}(\mathbf{V}^{p-1})^{ln} g_{nr}(\mathbf{V}^{e-1})^{rs} g_{sj} - (\mathbf{V}^e)_{im} g^{ml}(\mathbf{V}^{p-1})^{kn} g_{nr}(\mathbf{V}^{e-1})^{rs} g_{sj}). \tag{A4}$$

The inverse operator $\mathbb{A}^{p-1}$ is defined as solution of the linear algebraic equations

$$A^{p\,kl}_{ij}(\mathbb{A}^{p-1})^{mn}_{kl} = \delta^m_i \delta^n_j. \tag{A5}$$

The transposed operators are obtained by reordering the base vectors and the indices of the components in the reverse order.

B. DERIVATIVES OF $\ln \mathring{\mathbf{U}}^e$

The time derivative of the logarithmic strain tensor was investigated in Hoger (1987). The result obtained there is complex and not very appropriate for an implementation in a general solution algorithm. Besides, also some further properties of $(\ln \mathring{\mathbf{U}}^e)$ are needed here. Therefore, we have to reconsider the material time derivative of $\ln \mathring{\mathbf{U}}^e$ choosing a simpler way of derivation.

Let us split the strain rate tensor $\mathring{\mathbf{d}}^e$ according to (63) into a part $\mathring{\mathbf{d}}^e_c$, co-axial to $\mathring{\mathbf{U}}^e$, and into a part $\mathring{\mathbf{d}}^e_o$, off-axial to $\mathring{\mathbf{U}}^e$.

$$\begin{aligned} \mathring{\mathbf{d}}^e_c &= \mathring{\mathbf{d}}^e \cdot (\mathbf{e}_i \otimes \mathbf{e}_i)\mathbf{e}_i \otimes \mathbf{e}_i, \\ \mathring{\mathbf{d}}^e_o &= \mathring{\mathbf{d}}^e - \mathring{\mathbf{d}}^e_c. \end{aligned} \tag{B1}$$

where $\mathbf{e}_i$, $i \in \{1, 2, 3\}$, are the unit vectors in the principal directions and $\mathring{U}^e_i$ the principal values of $\mathring{\mathbf{U}}^e$ yielding the representation

$$\mathring{\mathbf{U}}^e = \sum_{i}^{3} \mathring{U}^e_i \mathbf{e}_i \otimes \mathbf{e}_i. \tag{B2}$$

Using eqn (63) we can show now that the off-axial part $\mathring{\mathbf{d}}^e_o$ corresponds to a pure rotation of the principal directions of $\mathring{\mathbf{U}}^e$ and $\ln \mathring{\mathbf{U}}^e$ and that the coaxial part $\mathring{\mathbf{d}}^e_c$ corresponds to a pure change of the principal values of them. From this one can derive

$$(\ln \mathring{\mathbf{U}}^e)^{\cdot} = \frac{\partial(\ln \mathring{\mathbf{U}}^e)}{\partial t}$$

$$= \mathring{\mathbf{d}}^e_c + 2\left[\sum_j \sum_{k \neq j} \left(\frac{\partial \dot{\mathring{\mathbf{U}}}{}^e}{\partial \mathring{\mathbf{d}}^e} \cdot \mathring{\mathbf{d}}^e\right) \cdot (\mathbf{e}_j \otimes \mathbf{e}_k) \frac{\mathbf{e}_j \otimes \mathbf{e}_k}{\mathring{U}^e_j - \mathring{U}^e_k} \ln \mathring{\mathbf{U}}^e\right]_{\rm sym}, \tag{B3}$$

where due to lack of space the details are not presented here. If there are two (or three) identical principal values of $\mathring{\mathbf{U}}^e$, the associated principal directions are undetermined and can be chosen so that there is no corresponding off-axial part of $\mathring{\mathbf{d}}^e$, which allows us to omit this part in the double sum of eqn (B3).
Using eqn (B3) one can immediately see that for all isotropic–elastic materials

$$\dot{W} = \mathring{\boldsymbol{\tau}} \cdot \mathring{\mathbf{d}}^e = \mathring{\boldsymbol{\tau}} \cdot (\ln \mathring{\mathbf{U}}^e)^{\cdot} \tag{B4}$$

holds, because there $\mathring{\boldsymbol{\tau}}$ is co-axial to $\mathring{\mathbf{U}}^e$ and thus only the first term of (B3), $\mathring{\mathbf{d}}^e_c$, contributes to the power (B4). Thus after push-forward with $\mathbf{Q}$ (see Section 6.1) the Kirchhoff stress tensor $\boldsymbol{\tau}$ is work-conjugate to the Eulerian logarithmic elastic strain tensor $\ln \mathbf{V}^e$, if the material is isotropic–elastic.

The operator $\partial(\ln \mathring{\mathbf{U}}^e)/\partial \mathring{\mathbf{d}}^e$, which is needed in Section 6.1, can be obtained from eqn (B3) as

$$\frac{\partial(\ln \mathring{\mathbf{U}}^e)}{\partial \mathring{\mathbf{d}}^e} = \sum_i \sum_j \left(\left.\frac{\partial(\ln \mathring{\mathbf{U}}^e)}{\partial t}\right|_{\mathring{\mathbf{d}}^e = (\mathbf{a}_i \otimes \mathbf{a}_j)_{\rm sym}}\right) \otimes \mathbf{a}_i \otimes \mathbf{a}_j, \tag{B5}$$

where $\mathbf{a}_i$, $i \in \{1, 2, 3\}$, are Cartesian base vectors.

Considering $\ln \mathring{\mathbf{U}}^e$ as a derivative of the formal potential $\Pi = \frac{1}{2}\ln \mathring{\mathbf{U}}^e \cdot \ln \mathring{\mathbf{U}}^e$ with $\ln \mathring{\mathbf{U}}^e = \partial \Pi / \partial \mathring{\mathbf{d}}^e$ the following symmetry condition

$$\left(\frac{\partial(\ln \mathring{\mathbf{U}}^e)}{\partial \mathring{\mathbf{d}}^e}\right)_{ijkl} = \left(\frac{\partial(\ln \mathring{\mathbf{U}}^e)}{\partial \mathring{\mathbf{d}}^e}\right)_{klij} \tag{B6}$$

can be proved. From the change of volume one can derive $\operatorname{tr} \mathring{\mathbf{d}}^e = \operatorname{tr}(\ln \mathring{\mathbf{U}}^e)^{\cdot}$ leading to

$$\sum_{i=1}^{3} \left(\frac{\partial(\ln \mathring{\mathbf{U}}^e)}{\partial \mathring{\mathbf{d}}^e}\right)_{iikk} = 1 \quad (\text{no sum over } k) \tag{B7}$$

and

$$\sum_{i=1}^{3} \left(\frac{\partial(\ln \mathring{\mathbf{U}}^e)}{\partial \mathring{\mathbf{d}}^e}\right)_{iijk} = 0 \quad \text{for} \quad j \neq k. \tag{B8}$$

C. DERIVATIVES OF ln $\mathbf{U}^p$

In Appendix B we derived the material time derivative of $\ln \mathring{\mathbf{U}}^e$ as function of $\mathring{\mathbf{d}}^e$. In this section we want to consider the material time derivative of $\ln \mathbf{U}^p$ as function of $\dot{\mathbf{U}}^p$. Analogously, we denote with $\dot{\mathbf{U}}^p_c$ that part of $\dot{\mathbf{U}}^p$, which is co-axial to $\mathbf{U}^p$, and with $\dot{\mathbf{U}}^p_o$ that part of $\dot{\mathbf{U}}^p$, which is off-axial to $\mathbf{U}^p$. They can be determined as follows

$$\dot{\mathbf{U}}^p_c = \dot{\mathbf{U}}^p \cdot (\mathbf{e}_i \otimes \mathbf{e}_i)\mathbf{e}_i \otimes \mathbf{e}_i,$$
$$\dot{\mathbf{U}}^p_o = \dot{\mathbf{U}}^p - \dot{\mathbf{U}}^p_c, \tag{C1}$$

where $\mathbf{e}_i$, $i \in \{1, 2, 3\}$ are the principal unit vectors of $\mathbf{U}^p$ with the associated principal values U^p_i leading to

$$\mathbf{U}^p = \sum_{i=1}^{3} U^p_i \mathbf{e}_i \otimes \mathbf{e}_i. \tag{C2}$$

Then the material time derivative of $\ln \mathbf{U}^p$ can be derived as

$$(\ln \mathbf{U}^p)^{\cdot} = \dot{\mathbf{U}}^p_c \mathbf{U}^{p\,-1} + 2\left[\sum_j \sum_{k \neq j} \dot{\mathbf{U}}^p \cdot (\mathbf{e}_j \otimes \mathbf{e}_k) \frac{\mathbf{e}_j \otimes \mathbf{e}_k}{U^p_j - U^p_k} \ln \mathbf{U}^p\right]_{\rm sym}. \tag{C3}$$

If there are two (or three) identical principal values of $\mathbf{U}^p$, the associated principal unit vectors are undetermined and can be chosen such that there is no corresponding off-axial part of $\dot{\mathbf{U}}^p$, which allows one to omit the corresponding part in the double sum of eqn (C3).

From eqn (C3) the operator $\partial(\ln \mathbf{U}^p)/\partial \dot{\mathbf{U}}^p$ can be obtained as

$$\frac{\partial(\ln \mathbf{U}^p)}{\partial \dot{\mathbf{U}}^p} = \sum_i \sum_j \left((\ln \mathbf{U}^p)^{\cdot}\big|_{\dot{\mathbf{U}}^p = (\mathbf{a}_i \otimes \mathbf{a}_j)_{\rm sym}}\right) \otimes \mathbf{a}_i \otimes \mathbf{a}_j, \tag{C4}$$

where $\mathbf{a}_i$, $i \in \{1, 2, 3\}$ are Cartesian base vectors.

The co-rotational rate of $\ln \mathbf{V}^p$, $(\ln \mathbf{V}^p)^{\nabla}$, and the operator $\partial(\ln \mathbf{V}^p)/\partial \mathbf{V}^p$ are the push-forwards with $\mathbf{Q}$ of eqns (C3) and (C4) resulting in a replacement of $\mathbf{U}^p$ and $\dot{\mathbf{U}}^p$ by $\mathbf{V}^p$ and $(\mathbf{V}^p)^{\nabla}$, respectively.